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Coherent states and energy spectrum of the anharmonic oscillator

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Abstract. The asymptotic behaviour of an energy level of the anharmonic oscillator with the anharmonicity λx^{2k} was obtained by the coherent states method.

1. Introduction

The anharmonic oscillator problem was studied by a number of authors in various aspects. Simon (1970), Bender (1970) and Bender and Wu (1969, 1973) have considered the analyticity properties of the energy level $E_n(\lambda)$ as a function of the coupling constant λ for the Hamiltonian $H=\frac{1}{2}(p^2+x^2)+\lambda x^4$. In particular, it has been shown in these works that the energy levels of the λx^4 anharmonic oscillator satisfy the condition

$$E_n(\lambda)\sim(n+\tfrac{1}{2})^{4/3}\lambda^{1/3}$$

for complex λ.

Within the framework of the quasiclassical approach Lakshmanan (1973), Lakshmanan and Prabhakaran (1973) and Mathews and Eswaran (1972) have found approximate formulae for the $\lambda x^4+\rho x^6$, ρx^6 and λx^4 anharmonic oscillators, respectively.

Numerical calculations of energy levels have been performed by Biswas *et al* (1971, 1973) for the anharmonic oscillator with anharmonicity $\lambda x^{2k}(0<\lambda\leqslant 50)$ and the energy level has been shown to be proportional to

$$\lambda^{1/(k+1)}(n+\tfrac{1}{2})^{2k/(k+1)}, \qquad n=0,1,2,\ldots,\ k=1,2,\ldots.$$

Halpern (1973) has considered the problem of the λx^4 anharmonic oscillator on the basis of non-linear canonical transformations of the eigenfunction basis of the corresponding Hamiltonian. The space of Fock–Bargmann representations has been used by Hioe and Montroll (1975) and Hioe *et al* (1976) to investigate the anharmonic potentials $x^2+\lambda x^4$ and $x^2+\lambda x^{2k}$.

The anharmonic oscillator problem is of interest both from the physical viewpoint (models of nuclear potentials, non-linear oscillator models of particles, etc) and from the pure mathematical one. We concentrate our attention on some mathematical aspects of this problem. The purpose of this work is to consider the λx^{2k} anharmonic oscillator on the basis of a combination of methods used by Halpern (1973), Hioe and Montroll (1975), Hioe *et al* (1976) and the coherent states method, which has been used widely in various fields of theoretical physics in recent years.

In § 2, we introduce a non-linear canonical transformation of the eigenfunctions for the usual harmonic oscillator Hamiltonian. This is done to find an appropriate basis formed by the transformed functions in which it is convenient to use perturbation theory. The unitary operator U of such a transformation can be defined and, as a result, the problem of calculating the matrix elements of the Hamiltonian H in the basis of the harmonic oscillator eigenfunctions reduces to a familiar problem, but for the transformed Hamiltonian $\tilde{H} = U^+HU$.

The coherent states method can be applied to the calculation of the matrix elements of the Hamiltonian U^+HU in first-order perturbation theory. Section 3 contains the definition of the coherent states of the one-dimensional harmonic oscillator and some of their properties are established (Glauber 1963, Man'ko 1972).

In § 4, we describe briefly the essence of the saddle-point method. This method enables the integrals contained in the corresponding expressions for the matrix elements to be calculated (Evgrafov 1968, Evgrafov and Postnikov 1970 and Doktorov *et al* 1975).

The matrix elements of the transformed Hamiltonian $\tilde{H}$ and the asymptotic expression for the energy levels of the λx^{2k} anharmonic oscillator are obtained in § 5. In this section, we also consider the particular cases of the anharmonic oscillators with the potentials $x^2+\lambda x^4$ and $x^2+\rho x^6$. The expressions for the energy levels obtained are shown to be analogous to those obtained by Halpern (1973), Lakshmanan and Prabhakaran (1973) and Mathews and Eswaran (1972).

2. Canonical transformation

The problem we consider is to obtain the solution of the eigenvalue problem

$$H^{(k)}\psi(x) = E\psi(x) \tag{2.1}$$

where

$$H^{(k)}(x, p) = \tfrac{1}{2}(p^2+x^2)+\lambda x^{2k} = H_0(x, p)+\lambda x^{2k} \tag{2.2a}$$

$$H_0(x, p) = \tfrac{1}{2}(p^2+x^2), \qquad p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x. \tag{2.2b}$$

For convenience we deal with the units system in which $\hbar = c = 1$, and for simplicity let mass and frequency be equal to unity: $m = \Omega = 1$. Let $\psi^0(x)$ be the eigenfunction for the Hamiltonian (2.2*b*) of the harmonic oscillator

$$H_0(x, p)\psi^0(x) = \epsilon\psi^0(x). \tag{2.3}$$

It is easy to check that the functions $\psi_n(x)$ and $\psi_n^0(x)$ satisfy the following asymptotic conditions for $x \to \infty$:

$$\psi_n(x) \sim \exp(-\alpha_k|x|^{k+1}), \qquad \alpha_k = \sqrt{2\lambda}(k+1)^{-1} \tag{2.4a}$$

$$\psi_n^0(x) \sim \exp(-\tfrac{1}{2}x^2). \tag{2.4b}$$

We must define here a unitary operator U and obtain a new complete orthonormal set of functions $\chi_n(x)$

$$\chi_n(x) = U\psi_n^0(x) \tag{2.5}$$

the asymptotic behaviour of which is the same as that of the functions $\psi_n(x)$. It has been shown (Halpern 1973) that such an operator exists and it can be determined by

the relation

$$\chi_n(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right)^{1/2}\psi_n^0(g(x)) \tag{2.6}$$

from which it follows by comparing (2.4*a*) and (2.4*b*) that:

$$f(x)\sim\begin{cases}O(x^{2/(k+1)}), & x\to\infty \qquad (2.7a)\\ O(|x|), & x\to 0 \qquad (2.7b)\end{cases}$$

where $f\equiv g^{-1}$ is the inverse function. The function $f(x)$ can be chosen in the form

$$f_{sm}(x)=[(1+\omega|x|^s)^{2/(k+1)}-1]^m \tag{2.8}$$

with the condition $sm=1$. That $f_{sm}(x)$ satisfies the relations (2.7*a*) and (2.7*b*) can be verified directly. Here ω is an arbitrary positive constant. The simplest form of the function $f_{sm}(x)$ seems to be obtained at $s=m=1$, i.e.

$$f(x)\equiv f_{11}(x)=(1+\omega|x|)^{2/(k+1)}-1. \tag{2.9}$$

(Note that Halpern (1973) has considered the case $s=2$, $m=\frac{1}{2}$ for the λx^4 anharmonic oscillator.) The function $f(x)$ (2.9) determines the operator U in equations (2.5) and (2.6) to within a factor $\omega>0$. Owing to the fact that the functions $\psi_n(x)$ and $\chi_n(x)=U\psi_n^0(x)$ are of the same asymptotic behaviour it appears more convenient to apply perturbation theory to the problem studied by using the set of the functions $\chi_n(x)$ rather than that of $\psi_n^0(x)$. It turns out that we must operate with the transformed Hamiltonian

$$\tilde{H}^{(k)}=U^+H^{(k)}U \tag{2.10}$$

in the old basis of the functions $\psi_n^0(x)$. The explicit form of the transformed Hamiltonian is analogous to that given by Halpern (1973) and it can be represented as follows:

$$\tilde{H}^{(k)}(x,p)=H_0^{(k)}(x,m)+L_k(x,p) \tag{2.11}$$

where

$$H_0^{(k)}(x,m)=(\mu(x))^{(2k-2)/(k+1)}(a+bx^2) \tag{2.12a}$$

$$L_k(x,p)=\lambda F_k(x)+\lambda(\mu(x))^{(2k-2)/(k+1)}(1+2\omega|x|)+\tfrac{1}{2}[(\mu(x))^{2/(k+1)}-1]^2$$
$$-\mathrm{i}\frac{k^2-1}{4\omega}\operatorname{sgn}x(\mu(x))^{(k-3)/(k+1)}p+\frac{(k-1)(5-k)}{32}(\mu(x))^{-4/(k+1)} \tag{2.12b}$$

and the following designations are used:

$$\mu(x)=1+\omega|x|$$
$$a=\frac{(k+1)^2}{4\omega^2}(m+\tfrac{1}{2}), \qquad b=\lambda\omega^2-\frac{(k+1)^2}{8\omega^2} \tag{2.13}$$

and

$$F_k(x)=\sum_{j=0}^{2k-1}\binom{2k-1}{j}(-1)^j(\mu(x))^{2j/(k+1)}. \tag{2.14}$$

In deriving equation (2.12*a*) the relation $p^2=(p^2+x^2)-x^2$ has been used and the Hamiltonian (2.2*b*) obtained has been replaced by its eigenvalue $(m+\frac{1}{2})$, because we

have dealt with the basis of its eigenfunctions $\psi_m^0(x)$. It can be proved easily that in the oscillator basis the matrix elements of the operators of type $x^a p^b$ are of the form $n^{(a+b)/2}$, where n is a quantum number (Halpern 1973). Hence, if we restrict ourselves only to first-order perturbation theory, we must operate with that part of the Hamiltonian (2.11) which is the maximum power of $\mu(x)$. This part is of the form (2.12a). The contribution of the matrix elements of the operator $L_k(x, p)$ to those of the operator $\tilde{H}^{(k)}$ is different from the contribution of the $H_0^{(k)}$ matrix elements by the factor n^{-1}, where n is a quantum number, and consequently the operator $L_k(x, p)$ can be ignored. Note that within the framework of the method suggested all the Hamiltonian $\tilde{H}^{(k)}$ (2.11) can be considered in the same way, when all orders of perturbation theory are used.

Finally, we wish to obtain the matrix elements

$$E_{nm}^{(k)} = \langle|\tilde{H}^{(k)}|m\rangle = \langle n|H_0^{(k)}(x, m)|m\rangle[1+O(m^{-1})], \qquad |n\rangle \equiv \psi_n^0(x) \tag{2.15}$$

where $H_0^{(k)}$ is of the form (1.12a) and $\psi_n^0(x)$ are the eigenfunctions of the harmonic oscillator Hamiltonian. The expression (2.15) can be transformed by using the coherent states method.

3. Coherent states of the harmonic oscillator

The functions of the coherent states $|\alpha\rangle$ are the eigenfunctions of the annihilation operator $a = 2^{-1/2}(x+ip)$

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{3.1}$$

where α is a complex number. The set of the functions $|\alpha\rangle$ forms the overcomplete non-orthogonal basis (Glauber 1963, Man'ko 1972). This means that there is some integral dependence between two functions $|\alpha\rangle$ and $|\beta\rangle$ (Glauber 1963). The basis non-orthogonality is determined by the inner product which can be chosen in the form

$$\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2).$$

The explicit expression of the function is of the form

$$|\alpha\rangle = \pi^{-1/4}\exp(-\tfrac{1}{2}\alpha^2 - \tfrac{1}{2}|\alpha|^2)\exp(-\tfrac{1}{2}x^2 + \alpha x\sqrt{2}). \tag{3.2}$$

According to Galuber (1963), the matrix elements of an operator M in the coherent basis $\langle\alpha|M|\beta\rangle$ can be expanded in a Taylor series in powers of α^* and β with the expansion coefficients

$$C_{nm} = (n!m!)^{-1/2}\langle n|M|m\rangle \tag{3.3}$$

where $|n\rangle \equiv \psi_n^0(x)$. The function

$$G(\alpha^*, \beta) = \langle\alpha|M|\beta\rangle\exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2) \tag{3.4}$$

can easily be proved to be the analytic function of two variables α^* and β (Glauber 1963). Hence we can use the well known integral formula for the Taylor series coefficients:

$$C_{nm} = \left(\frac{1}{2\pi i}\right)^2 \int_c d\alpha^*\, d\beta\, (\alpha^*)^{-n-1}\beta^{-m-1}G(\alpha^*, \beta) \tag{3.5}$$

where c is the contour of integration which goes round the origin of the coordinate

system. We use the operator $H_0^{(k)}$ (2.12a) instead of the operator M in equations (3.3)–(3.5) and we obtain an expression for the matrix elements $E_{nm}^{(k)}$ determined by the relation (2.15):

$$E_{nm}^{(k)} = -\frac{\sqrt{n!m!}}{4\pi^2}\int_c \mathrm{d}\alpha^* \,\mathrm{d}\beta\, (\alpha^*)^{-n-1}\beta^{-m-1}G(\alpha^*, \beta)[1+\mathrm{O}(m^{-1})] \tag{3.6}$$

where

$$G(\alpha^*, \beta) = \langle\alpha|H_0^{(k)}(x, m)|\beta\rangle \exp(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2). \tag{3.7}$$

Since there are quadratic exponents in the integrands (3.6), all the integrals in x, α^* and β converge and the sequence of integrations can be altered. First, the integrals in α^* and β have to be calculated by an asymptotic method.

4. The saddle-point method

The saddle-point method (Evgrafov and Postnikov 1968, Doktorov *et al* 1975) is used to calculate integrals of the form

$$\Phi(l) = \int_c \phi(z)\exp(lf(z))\,\mathrm{d}z, \qquad z = (z_1, z_2, \ldots, z_p) \tag{4.1}$$

where l is a large complex number, p is the space dimension, c is a contour of integration the ends of which do not contribute significantly to the integral, $\phi(z)$ and $f(z)$ are the analytic functions. In fact, this method consists of two steps. First, the contour of integration c is deformed in an appropriate way and the new contour $\tilde{c}$ is such that on it the condition $\mathrm{Im}\, f(z) \equiv 0$ is satisfied. The contributions from a set of saddle points can then be found by the usual Laplace method (Evgrafov 1968). The saddle-point method for n-dimensional integrals has been developed by Evgrafov and Postnikov (1970). Finally, we formulate the conditions which the function $f(z)$ must satisfy:

$$\begin{array}{ll} \text{(i)} & \mathrm{grad}\, f(z) = 0 \\ \text{(ii)} & \mathrm{Re}[\mathrm{d}^2 f(z)]_{z=z_0} < 0 \\ \text{(iii)} & \mathrm{He}\, f(z_0) \neq 0 \end{array} \tag{4.2}$$

for each saddle point z_0. Here $\mathrm{He}\, f(z_0)$ is the Hessian of the function $f(z)$ at the saddle point z_0:

$$\mathrm{He}\, f(z_0) = \det\left|\frac{\partial^2 f}{\partial z_i\, \partial z_j}\right|_{z=z_0}, \qquad i, j = 1, 2, \ldots, p. \tag{4.3}$$

It is also assumed that $\phi(z)$ changes sufficiently slowly near z_0. If the conditions (4.2) are satisfied, the general contribution of each saddle point to the integral (4.1) will be of the form

$$\mathcal{I}(l) = (-2\pi/l)^{p/2}\exp(lf(z_0))(\mathrm{He}\, f(z_0))^{-1/2}\phi(z_0). \tag{4.4}$$

More specifically

$$\Phi(l) = \mathcal{I}(l)[1+\mathrm{O}(l^{-1})]. \tag{4.5}$$

The quantity $\mathcal{I}(l)$ (4.4) is the first term of an asymptotic series and $\mathcal{I}(l)$ is valid for large values of l. For our purpose, only the first term $\mathcal{I}(l)$ must be calculated, as follows from equation (4.5).

5. Matrix elements

Let us apply the saddle-point method to the calculation of the matrix elements of the form (3.6). Combining equations (3.6) and (3.7) the formula for the matrix elements can be rewritten as follows:

$$E_{nm}^{(k)} = \left(\frac{n!m!}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}x\, I_{nm}(x)\, \mathrm{e}^{-x^2} H_0^{(k)}(x, m)[1+\mathrm{O}(m^{-1})] \tag{5.1}$$

where the two-dimensional integral in α^* and β is equal to

$$I_{nm}(x) = \left(\frac{1}{2\pi i}\right)^2 \int_c \frac{\mathrm{d}\alpha^*\, \mathrm{d}\beta}{\alpha^*\beta} \exp\left(-\frac{(\alpha^*)^2}{2} - \frac{\beta^2}{2} - n \ln \alpha^* - m \ln \beta\right) \tag{5.2}$$

and the Hamiltonian $H_0^{(k)}(x, m)$ given by the relationship (2.12*a*) does not depend on the momentum operator p. Using (4.2) we obtain for the function

$$f(\alpha^*, \beta) = -\tfrac{1}{2}(\alpha^*)^2 - \tfrac{1}{2}\beta^2 - n \ln \alpha^* - m \ln \beta \tag{5.3}$$

two sets of the saddle points

$$\alpha_{1,2} = \sqrt{n} \exp(\pm i\phi_n(x)) \qquad \beta_{1,2} = \sqrt{m} \exp(\pm i\phi_m(x)) \tag{5.4}$$

where

$$\cos \phi_s(x) = x(2s)^{-1/2}, \qquad s = n, m$$
$$|x| < \sqrt{2 \min(n, m)}$$

and when the case $p = 2$ in equation (4.4) is used, the term $\mathcal{J}(l)$ is equal to the sum $I_{nm}(x)$ of the contributions from two sets $(\alpha_{1,2}^*, \beta_{1,2})$ of saddle points, i.e. it follows from formula (5.2) that

$$I_{nm}(x) = (\pi n!\, m!)^{-1/2}\, \mathrm{e}^{x^2} \cos(\xi_{nm}(x))[(2n-x^2)(2m-x^2)]^{-1/4}\left[1+\mathrm{O}\left(\frac{1}{\min(n, m)}\right)\right] \tag{5.5}$$

where

$$\xi_{nm}(x) = \tfrac{1}{2}x[\sqrt{2m-x^2} - \sqrt{2n-x^2} - (m+\tfrac{1}{2})\phi_m(x) + (n+\tfrac{1}{2})\phi_n(x)]. \tag{5.6}$$

We have obtained formula (5.5) by using the function $lf(z) = f(\alpha^*, \beta)$ (5.3) in equations (4.3)–(4.5). The well known Stirling formula has also been used:

$$n! \simeq \sqrt{2\pi n}\, \mathrm{e}^{-n} n^n [1+\mathrm{O}(n^{-1})].$$

Putting $x < -\sqrt{2 \max(n, m)}$ or $x > +\sqrt{2 \max(n, m)}$, from the conditions (4.2) we find that there are three sets of saddle points, namely

$$\alpha_{3,4} = \sqrt{n} \exp(-\tilde{\phi}_n(-x) \pm i\pi) \qquad \beta_{3,4} = \sqrt{m} \exp(-\tilde{\phi}_m(-x) \pm i\pi) \tag{5.7}$$

for the interval $x < -\sqrt{2 \max(n, m)}$ and

$$\alpha_5 = \sqrt{n} \exp(-\tilde{\phi}_n(x)) \qquad \beta_5 = \sqrt{m} \exp(-\tilde{\phi}_m(x)) \tag{5.8}$$

for the interval $x > \sqrt{2 \max(n, m)}$. Here $\cosh \tilde{\phi}_s(x) = x(2s)^{-1/2}$, $s = n, m$. We obtain by comparison of the relations (5.4), (5.7) and (5.8) that, first, the points $|x| = \sqrt{2 \min(n, m)}$ divide the x axis into an oscillating region at $|x| \leqslant \sqrt{2 \min(n, m)}$ and exponentially decreasing regions if $|x| > \sqrt{2 \min(n, m)}$ and, second, the points

$|x| = \sqrt{2\max(n, m)}$ divide the latter regions into pure exponentially decreasing subregions in which there are only saddle points of the forms (5.7) or (5.8) $(-\infty < x < -\sqrt{2\max(n, m)}$, $\sqrt{2\max(n, m)} < x < +\infty)$, and subregions in which there are saddle points of the form (5.4) and those of the forms (5.7) or (5.8) $(-\sqrt{2\max(n, m)} \leq x < -\sqrt{2\min(n, m)}$, $\sqrt{2\min(n, m)} < x < \sqrt{2 max(n, m)})$.

It is easy to show, by inspection of the sets of saddle points (5.7) and (5.8) in the exponentially decreasing regions, that the contributions from these saddle points are exponentially small and can be ignored. Finally, only the term $I_{nm}(x)$ (5.5) gives a substantial contribution to the expression for the matrix elements (5.1). This formula can be rewritten by using the explicit form of the $I_{nm}(x)$ (5.5):

$$E_{nm}^{(k)} = \frac{1}{\pi}\int_{-\sqrt{2\min(n,m)}}^{+\sqrt{2\min(n,m)}} dx\, H_0^{(k)}(x, m)\cos(\xi_{nm}(x)) \times [(2n-x^2)(2m-x^2)]^{-1/4}\left[1+O\left(\frac{1}{\min(n, m)}\right)\right] \tag{5.9}$$

where $\xi_{nm}(x)$ and $H_0^{(k)}(x, m)$ are given by equations (5.6) and (2.12*a*) respectively.

Equation (5.9) can be derived by using the Maclaurin series of the functions $\cos^{-1}(x(2s)^{-1/2})$, $s = n, m$, contained in $\xi_{nm}(x)$ when only two first terms in each series are taken into account. The two first terms provide us with the required accuracy. We obtain

$$E_{nm}^{(k)} = \frac{1}{\pi}\int_{-\sqrt{2\min(n,m)}}^{+\sqrt{2\min(n,m)}} dx\, H_0^{(k)}(x, m)\kappa_{nm}(x) \times [(2n-x^2)(2m-x^2)]^{-1/4}\left[1+O\left(\frac{1}{\min(n, m)}\right)\right] \tag{5.10}$$

where

$$\kappa_{nm}(x) = \begin{cases} (-1)^{(m-n)/2}\cos\gamma_{nm}(x), & m-n = 2s \\ (-1)^{(m-n-1)/2}\sin\gamma_{nm}(x), & m-n = 2s+1,\ s = 0, \pm1, \pm2, \ldots \end{cases} \tag{5.11}$$

and

$$\gamma_{nm}(x) \approx \frac{x}{2}\left(\sqrt{2m-x^2} - \sqrt{2n-x^2} + \frac{m+\frac{1}{2}}{\sqrt{2m}} - \frac{n+\frac{1}{2}}{\sqrt{2n}}\right). \tag{5.12}$$

Using the parity properties

$$H_0^{(k)}(-x, m) = H_0^{(k)}(x, m) \qquad \text{and} \qquad \gamma_{nm}(-x) = -\gamma_{nm}(x)$$

and choosing the second form of the function $\kappa_{nm}(x)$:

$$\kappa_{nm}(x) = (-1)^{(m-n-1)/2}\sin\gamma_{nm}(x)$$

we obtain immediately that the odd matrix elements are identically equal to zero:

$$E_{n,n+2s+1}^{(k)} \equiv 0, \qquad s = 0, \pm1, \pm2, \ldots. \tag{5.13}$$

This relation corresponds to the fact that the original Hamiltonian (2.2*a*), the transformation function $f(x)$ (2.9) and the oscillator functions $\psi_n^0(x)$ are even functions with respect to the variable x.

It has been shown by Biswas *et al* (1973) that the energy level $E_n^{(k)}$ of the λx^{2k} anharmonic oscillator is proportional to the term $n^{2k/(k+1)}$, i.e. $E_n^{(k)} = C_k(\lambda)(n+\frac{1}{2})^{2k/(k+1)}$, where n is a quantum number, $C_k(\lambda)$ is a function of variable λ. Let us consider the matrix elements with $n \gg m$ (the opposite case is reduced to the former by the reciprocal permutation of the indices n and m). Using the above remark on the form of the energy level $E_n^{(k)}$ and the integral representation of the $B(\mu, \nu)$ function (Gradsteyn and Ryzhik 1963), we obtain an approximate expression for the matrix elements by calculation of the integral in (5.10) ($n \gg m$):

$$|E_{nm}^{(k)}| \leqslant \left(\frac{m}{n}\right)^{1/4} m^{2k/(k+1)} C_k(\lambda, \omega) = \left(\frac{m}{n}\right)^{1/4} E_{mm}^{(k)}(\omega) \tag{5.14}$$

where

$$C_k(\lambda, \omega) = \pi^{-1}(\omega\sqrt{2})^{(2k-2)/(k+1)} B\left(\frac{3k-1}{2k+2}, \frac{3}{4}\right)\left(a_0 + 4b\frac{3k-1}{9k+1}\right)$$

$$a_0 = \frac{(k+1)^2}{4\omega^2}.$$

The correlation (5.14) defines the law of decrease of the matrix elements. It shows that $|E_{nm}^{(k)}| \to 0$ as the term $(m/n)^{1/4}$ with the condition $m < n$, $n \to \infty$.

As can be seen from (5.14) at $n \sim m$, the terms $E_{nm}^{(k)}$ are comparable with the diagonal elements, and the Hamiltonian matrix is not diagonal. The results we obtain in this case are of the same accuracy as those obtained by Halpern (1973) for $k = 2$. Finally, the expression for the energy levels is obtained in first-order perturbation theory by putting $n = m$ in formula (5.10), i.e. we consider the diagonal matrix elements

$$E_n^{(k)} = \frac{2}{\pi}\int_0^{\sqrt{2n}} dx\, H_0^{(k)}(x, n)(2n - x^2)^{-1/2}[1 + O(n^{-1})]. \tag{5.15}$$

This relation can be rewritten by changing the variable $x = y\sqrt{2n}$ as follows

$$E_n^{(k)} = \frac{2n}{\pi}\int_0^1 dx\, (1-x^2)^{-1/2}(1+\epsilon_n x)^s (a_0 + 2bx^2)[1 + O(n^{-1})] \tag{5.16}$$

where the explicit form of $H_0^{(k)}$ (2.12a) and the following designations are used:

$$\epsilon_n = \omega\sqrt{2n}, \qquad s = \frac{2k-2}{k+1}$$
$$a_0 = \frac{(k+1)^2}{4\omega^2}, \qquad b = \lambda\omega^2 - \frac{(k+1)^2}{8\omega^2}. \tag{5.17}$$

We also put $n + \frac{1}{2} \approx n$. Integration in the relation (5.16) can be performed directly, and the result of the integration has to be expressed in terms of the hypergeometric function of two variables (Gradsteyn and Ryzhik 1963). But a more detailed inspection of the properties of the hypergeometric function shows that we can use the approximation $(1+\epsilon_n x)^s \approx \epsilon_n^s x^s$, which, when applied, simplifies the integration of (5.16). The integral is then reduced to

$$\pi^{-1}(2n)^{1+\frac{1}{2}s}\omega^s \int_0^1 dx\, (1-x^2)^{-1/2} x^s (a_0 + 2bx^2)$$

and after the integrations (Gradsteyn and Ryzhik 1963) we obtain

$$E_n^{(k)} = \pi^{-1/2}(2n)^{1+\frac{1}{2}s}\omega^s\Gamma\left(\frac{1+s}{2}\right)\Gamma^{-1}\left(1+\frac{s}{2}\right)\left(a_0+2b\frac{s+1}{s+2}\right)[1+O(n^{-1})].$$

Using the designations (5.17) and the well known relation $z\Gamma(z)=\Gamma(z+1)$ for the Γ functions, the matrix elements are expressed as follows:

$$E_n^{(k)} = \pi^{-1/2}2^{-(2k+4)/(k+1)}(k+1)[k(3k-1)]^{-1}\Gamma\left(\frac{5k+1}{2k+2}\right)\Gamma^{-1}\left(\frac{2k}{k+1}\right)$$
$$\times[8(3k-1)\lambda\omega^{4k(k+1)}+(k+1)^3\omega^{-4(k+1)}](n+\tfrac{1}{2})^{2k/(k+1)}[1+O(n^{-1})] \tag{5.18}$$

where again we have used the approximate relation $n \approx n+\frac{1}{2}$. The term $E_n^{(k)}(\omega)$ determined by formula (5.18) is the function of the parameter ω. If we use the 'stationary' condition $\partial E_n^{(k)}(\omega)/\partial\omega = 0$ we obtain a fixed ω_0 in the form

$$\omega_0 = (k+1)^{3/4}[8k\lambda(3k-1)]^{-1/4}. \tag{5.19}$$

The final expression for the matrix elements is obtained by substituting (5.19) into formula (5.18):

$$E_n^{(k)} = \pi^{-1/2}[(k+1)^{5k+2}(2k)^{-2k-1}(3k-1)^{-k}]^{1/(k+1)}\Gamma\left(\frac{5k+1}{2k+2}\right)$$
$$\times\Gamma^{-1}\left(\frac{2k}{k+1}\right)(n+\tfrac{1}{2})^{2k/(k+1)}\lambda^{1/(k+1)}[1+O(n^{-1})]. \tag{5.20}$$

We are now able to obtain formulae analogous to those derived by Halpern (1973), Mathews and Eswaran (1972) and Lakshmanan and Prabhakaran (1973).

First, in relation (5.18) let k be equal to two, the result obtained coincides with the formula for the diagonal elements obtained by Halpern (1973):

$$E_n^{(2)}(\omega) = \frac{2^{1/3}\Gamma(\frac{5}{6})}{24\sqrt{\pi}\Gamma(\frac{7}{3})}(n+\tfrac{1}{2})^{4/3}(40\lambda\omega^{8/3}+27\omega^{-4/3})[1+O(n^{-1})] \tag{5.21}$$

where ω = constant > 0.

Analogous formulae (for $k = 2, 3$) can be obtained from relation (5.20). For $k = 2$, it follows that

$$E_n^{(2)}(\omega_0) = \frac{81(20)^{1/3}\Gamma(\frac{11}{6})}{80\sqrt{\pi}\Gamma(\frac{4}{3})}(n+\tfrac{1}{2})^{4/3}\lambda^{1/3}[1+O(n^{-1})] \tag{5.22}$$

and for $k = 3$ we obtain

$$E_n^{(3)}(\omega_0) = \pi^{-1}\tfrac{32}{3}(\tfrac{4}{3})^{1/4}(n+\tfrac{1}{2})^{3/2}\lambda^{1/4}[1+O(n^{-1})]. \tag{5.23}$$

These results are similar to those obtained by Mathews and Eswaran (1972) and Lakshmanan and Prabhakaran (1973).

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